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BRIEF COMMUNICATION

Moments of powers of the Hulthén density

M. L. Glasser · I. Nagy

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1 Introduction

The Hulthén Hamiltonian

$$H = -\frac{1}{2}\nabla - Z\frac{\Lambda}{e^{\Lambda r} - 1} \tag{1}$$

is exactly solvable with normalized ground-state wave function

$$\phi_0(r) = N_0 \frac{1}{\Lambda r}\left[e^{-(Z-\Lambda/2)r} - e^{-(Z+\Lambda/2)r}\right]$$
$$N_0 = \left[\frac{Z(4Z^2 - \Lambda^2)}{4\pi}\right]^{1/2} \tag{2}$$

and corresponding energy

$$E_0(\Lambda) = -\frac{Z^2}{2}\left(1 - \frac{\Lambda}{2Z}\right)^2 \tag{3}$$

M. L. Glasser (✉)
Department of Physics, Clarkson University, Potsdam, NY 13699-5820, USA
e-mail: laryg@clarkson.edu

M. L. Glasser · I. Nagy
Donostia International Physics Center, P. Manuel de Lardizabal 4, 20018 San Sebastian, Spain

I. Nagy
Department of Theoretical Physics, Institute of Physics, Technical University of Budapest,
1521 Budapest, Hungary

where $\Lambda \leq 2Z$ is a freely chosen screening parameter. For many years it has been used as a model for treating a variety of phenomena in nuclear [1,2] and Condensed matter physics [3–5]. In these applications one frequently requires various moment integrals, frequently of the density $n(r) = \phi_0(r)^2$

$$M(s, \nu) = \int_0^\infty r^{s-1}\phi_0(r)^\nu dr. \tag{4}$$

The aim of this note is to present the exact evaluation of the class $M(m+\nu, \nu)$ where m is a non-negative integer and $\nu > 0$.

2 Evaluation

We begin by scaling one of the exponents out of the integrals through the substitution $x = (Z + \Lambda/2)r$ and introducing the parameter $a = (Z - \Lambda/2)/(Z + \Lambda/2)$ to obtain

$$\begin{aligned} M(s, \nu) &= \frac{N_0^\nu}{\Lambda^\nu (Z + \Lambda/2)^s} I(s, \nu, a) \\ I(s, \nu, a) &= \int_0^\infty x^{s-\nu-1} e^{-a\nu x}(1 - e^{-(1-a)x})^\nu dx \end{aligned} \tag{5}$$

where the leading exponential has been factored out of the parenthesis. We next apply the binomial theorem to obtain

$$I(s, \nu, a) = \sum_{k=0}^{\infty} (-1)^k \binom{\nu}{k} \int_0^\infty x^{s-\nu-1} e^{-[a\nu + (1-a)k]x} dx \tag{6}$$

The integral in (6) is Gauss' Gamma function integral and the binomial coefficient can be expressed in terms of Polchammer's factorial symbol

$$\binom{\nu}{k} = \frac{(-1)^k}{k!}(-\nu)_k$$

to yield

$$I(s, \nu, a) = \frac{\Gamma(s-\nu)}{(1-a)^{s-\nu}} \sum_{k=0}^{\infty} \frac{(-\nu)_k}{k!} \frac{1}{(k+b)^{s-\nu}} \tag{7}$$

where $b = a\nu/(1-a)$. There are several cases in which the series in (7) can be expressed in closed form.

2.1 Case 1: $s = m + \nu$

Here, when $m = 1, 2, 3, \vdots$ we have

$$I(m+\nu,\nu,a) = \frac{(m-1)!}{(1-a)^m}\sum_{k=0}^{\infty}\frac{(-\nu)_k}{k!}\left[\frac{\Gamma(k+b)}{\Gamma(k+b+1)}\right]^m \tag{8}$$

and since the ratio of Gamma functions is $(b)_k/b(b+1)_k$, we have, by definition,

$$I(m+\nu,\nu,a) = \frac{(m-1)!}{(a\nu)^m}\ {}_{m+1}F_m(-\nu, b,\ldots,b; b+1,\ldots,b+1; 1) \tag{9}$$

in terms of the generalized hypergeometric function [6]. For the case of unit argument, as here, this hypergeometric function can be summed by the formula

$$ {}_{p+1}F_p(\alpha,\beta_1,\ldots,\beta_p;\beta_1+1,\ldots,\beta_p+1;z) = \sum_{k=1}^{p}\ {}_2F_1(\alpha,\beta_k;\beta_k+1;z) \prod_{l=1}^{p}\frac{\beta_l}{\beta_l-\beta_k} \tag{10}$$

where the factor with $l = k$ is omitted from the product. Now, ${}_2F_1(\alpha,\beta;\beta+1;1) = \Gamma(1-a)\Gamma(\beta+1)/\Gamma(\beta-\alpha+1)$ and the case where the beta's become equal can be handled by L'Hospital's rule. Without much difficulty, we find

$$I(m+\nu,\nu,a) = \frac{(-1)^{m-1}}{(1-a)^m}\Gamma(\nu+1)\frac{d^{m-1}}{dx^{m-1}}\frac{\Gamma(x)}{\Gamma(x+\nu+1)}|_{x=b}. \tag{11}$$

2.2 Case 2: $m = 0$

Here we deal with the integral

$$I(\nu,\nu,a) = \int_0^{\infty}\frac{dx}{x}e^{-a\nu x}(1-e^{-(1-a)x})^{\nu}. \tag{12}$$

First we note the Laplace transform [7]

$$\int_0^{\infty} dt e^{-pt}(1-e^{-(1-a)t})^{\nu} = \frac{\Gamma\left(\frac{p}{1-a}\right)\Gamma(\nu+1)}{(1-a)\Gamma\left(\frac{p}{1-a}+\nu+1\right)} \tag{13}$$

and the inverse Laplace transform [7]

$$\int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i t} e^{(p-a\nu)t} = \theta(p - a\nu). \tag{14}$$

Then, by the Parseval relation for the Laplace transform

$$\begin{aligned} I(\nu, \nu, a) &= \frac{\Gamma(\nu+1)}{1-a} \int_{a\nu}^{\infty} \frac{\Gamma\left(\frac{p}{a-1}\right)}{\Gamma\left(\frac{p}{a-1} + \nu + 1\right)} dp \\ &= \Gamma(\nu+1) \int_{b}^{\infty} \frac{\Gamma(x)}{\Gamma(x+\nu+1)} dx \end{aligned} \tag{15}$$

In general one can get no further analytically; however, when ν is a positive integer

$$\begin{aligned} I(n, n, a) &= n! \int_{b}^{\infty} \frac{dx}{x(x+1)\ldots(x+n)} \\ &= \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \ln\left(\frac{na}{1-a} + k\right) \end{aligned} \tag{16}$$

by partial fractions.

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References

1. L. Hulthén, M. Sugawara, ed. by S. Pflügge Handbuch der Physik XXXIX (Springer, Berlin, 1937)
2. Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954)
3. P.F. Meier, Helv. Physica Acta **48**, 227 (1975)
4. I. Nagy, M.L. Glasser, N.H. March, Phys. Lett. A **373**, 3182 (2009)
5. I. Nagy, Phys. Rev. A **83**, 044503 (2011)
6. A.P. Prudikov et al. *Integrals and Series*, Chapter 7, (Gordon and Breach, New York, 1990)
7. A. Érdélyi et al., *Tables of Integral Transforms, Vol. 1* (McGraw-Hill, New York, 1953)